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Quantization of set theory and generalization of the fermion algebra

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Abstract
The quantum states of a d-dimensional fermion algebra are in one to one correspondence with the subsets of a d-element universal set. In this paper we use this set theoretical motivation to construct a one-parameter deformation of the fermion algebra and extend it to a d-dimensional generalization which is invariant under the group $U(d)$. This discrete fermionic oscillator system is extended to the continuous case. We also show that the q-deformation of these systems is related to supercovariant q-oscillators.

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1. Introduction

The importance of the single fermion algebra defined by the relations

$$cc^* + c^*c = 1 \qquad c^2 = 0 \tag{1}$$

is well known in various branches of physics. This algebra has a unique two-dimensional representation which corresponds to the vacuum and one-particle states of the single fermion.

The d-dimensional generalization,

$$c_i c_j^* + c_j^* c_i = \delta_{ij} \qquad c_i c_j + c_j c_i = 0 \tag{2}$$

enjoys a $U(d)$ symmetry which acts on the annihilation operator c_i by

$$c_i \to \sum_j u_{ij} c_j \tag{3}$$

where the complex matrix $\{u_{ij}\}$ is unitary. A nontrivial extension of this algebra which generalizes (2) is widely used in field theory and entails replacement of the discrete indices

i, j to continuous indices p, q together with replacement of the Kronecker δ_{ij} with the Dirac $\delta(p-q)$ and reads

$$\begin{aligned} c(p)c^*(q) + c^*(q)c(p) &= \delta(p-q) \\ c(p)c(q) + c(q)c(p) &= 0. \end{aligned} \tag{4}$$

Not all fermion-like algebras enjoy this generalization. The most well-known example being

$$c_i c_i^* + c_i^* c_i = 1 \qquad c_i^2 = 0 \qquad c_i c_j = c_j c_i \qquad c_i c_j^* = c_j^* c_i \qquad i \neq j. \tag{5}$$

This algebra just like (2) has a 2^d-dimensional representation and reduces to (1) for the one-dimensional case but does not enjoy $U(d)$ symmetry and cannot be consistently generalized to continuous indices in the manner that (2) can be generalized to (4). Physically, it can be said that, due to the condition $c_i^2 = 0$, (5) obeys the Pauli exclusion principle which says that two identical fermions cannot be put into the same state. However, it does not obey the anti-symmetry of the wavefunction under fermion interchange.

Various q-deformations of the fermion algebra (1) and its supersymmetric generalizations have been investigated [1–12]. The multidimensional version of q-deformed fermion algebra [12] cannot be generalized to be the continuous case as in (4) in a smooth manner. In some sense all one-dimensional generalized (deformed) fermion algebras are alike [8] since the physical requirement of the vacuum and one-particle states gives rise to a two-dimensional algebra and thus any element of this algebra can be expressed in the form

$$a = \alpha c + \beta c^* + \gamma c^* c + \delta c c^* \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C} \tag{6}$$

where c and c^* are the conventional fermion creation and annihilation operators and $\mathbb{C}$ indicates complex numbers. However for multidimensional and especially continuous generalizations such as (4), deformation will be nontrivial since not all fermion algebras, e.g. (5), can be made continuous. In this paper, we will investigate a fermion algebra whose continuous version reads

$$\begin{aligned} a(p)a^*(q) + a^*(q)a(p) &= s(p)a^*(q) + \overline{s(q)}a(p) + \delta(p-q) - s(p)\overline{s(q)} \\ a(p)a(q) + a(q)a(p) &= s(p)a(q) + s(q)a(p) \end{aligned} \tag{7}$$

where $s(p)$ is a deformation function satisfying

$$\int_{-\infty}^{\infty} |s(p)|^2 \, \mathrm{d}p < 1. \tag{8}$$

For $s(p) \equiv 0$, the undeformed, continuous fermion commutation relations (4) are obtained.

In section 2, we will construct the one-dimensional generalized fermion algebra and find its unique representation. In section 3, we will present the multidimensional and the continuous extensions. Section 4 discusses a q-deformation of the generalized fermion algebra which turns out to be related to a supersymmetric system of q-bosons and q-fermions. Section 5 is reserved for conclusions.

2. The generalized fermion algebra

The motivation leading to (7) involves several steps. The first step is the observation that the concept of a state containing at most one fermion is similar to the concept of a set at most containing one of a given element. We start by considering a Hilbert space formulation of sets. We assume every subset of a given finite universal set to be a normalizable vector in a Hilbert space. We start our construction with an operator a_i^* which we identify with the union operator which unites the ith element of the universal set to a given set which will be described by a vector in a Hilbert space. We consider a_i^* for fixed i and omit the index i. Since uniting the same element any nonzero number of times does not change the set, a^* should satisfy

$$a^{*2} = \bar{s}a^* \tag{9}$$

where s is a complex number. The case $s = 0$ gives the fermion algebra. For this case the creation operator a^* almost behaves like an operator which unites an element of the universal set with a given set. However, due to the fermionic nature of a^*, applying it twice gives zero whereas in set theory adding an element to the set any number of times does not change the set. Thus we are led to consider that $s \neq 0$. We will now show that up to a rescaling of a and a^*, (9) implies

$$aa^* = -a^*a + \bar{s}a + sa^* + 1 - s\bar{s}. \tag{10}$$

The normalization is chosen such that a^*a has eigenvalues 0 and 1 so that it can be identified with the fermionic number operator. To prove (10) let

$$aa^* + a^*a - \bar{s}a - sa^* = D. \tag{11}$$

Then one readily computes that D is central, i.e. in a representation D will be represented by a multiple of the identity. Let us denote the eigenvector of a^*a with the eigenvalue λ by $|\lambda\rangle$. It follows that $a^*|\lambda\rangle$ is also an eigenvector corresponding to the eigenvalue $s\bar{s} + D$. Then

$$(a^*a)^2|\lambda\rangle = \lambda^2|\lambda\rangle \tag{12}$$

$$a^*(aa^*a)|\lambda\rangle = a^*(s\bar{s} + D)a|\lambda\rangle = \lambda(s\bar{s} + D)|\lambda\rangle. \tag{13}$$

Comparing (12) and (13) it follows that

$$\lambda(\lambda - s\bar{s} - D) = 0. \tag{14}$$

Therefore, eigenvalues of a^*a can either be 0 or $s\bar{s} + D$. For a nontrivial solution $s\bar{s} + D > 0$, by rescaling the operators a, a^* and the complex number s, we may set

$$aa^* = -a^*a + \bar{s}a + sa^* + 1 - s\bar{s} \qquad a^2 = sa \tag{15}$$

so that a^*a has eigenvalue 0 or 1. It also follows that

$$aa^*a = a. \tag{16}$$

Applying (16) to $|\lambda\rangle$, we have for $\lambda \neq 1$

$$a|0\rangle = 0. \tag{17}$$

From (9), multiplying on the right by a and applying the resulting equation to $|\lambda\rangle$, it follows that

$$a^*|1\rangle = \bar{s}|1\rangle. \tag{18}$$

Taking the Hermitian conjugate of (16) and applying this to $|\lambda\rangle$, it follows that

$$a^*|0\rangle = \alpha|1\rangle \tag{19}$$

so that

$$\langle 0|aa^*|0\rangle = \alpha\bar{\alpha}\langle 1|1\rangle. \tag{20}$$

If we replace aa^* in (20) by (15) and use that $|0\rangle, |1\rangle$ are orthonormal, the result gives $\alpha\bar{\alpha} = 1 - s\bar{s}$, hence $0 < |s| < 1$. The phase of α can be absorbed into the definition of $|0\rangle$.

Applying (15) on $|1\rangle$

$$a|1\rangle = s|1\rangle + \sqrt{1-s\bar{s}}|0\rangle. \tag{21}$$

Hence the actions of the operators a, a^* are given by (17), (18), (19) and (21) where $\alpha = \sqrt{1-s\bar{s}}$. Since the nontrivial representation of the algebra (15) for $|s| < 1$ is two-dimensional, we can consider this algebra as a deformation of the fermion algebra.

Taking the orthonormal vectors $|0\rangle$ and $|1\rangle$ to be the standard basis, the matrices of the operators a, a^*a are given by

$$a = \begin{pmatrix} 0 & \sqrt{1-s\bar{s}} \\ 0 & s \end{pmatrix} \qquad a^*a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Note that for $s \neq 0$, aa^* is not diagonal in this basis since a^*a and aa^* do not commute. We will name the algebra characterized by the commutation relations (15), where $0 < |s| < 1$ as the quasi-fermion algebra. As we have shown, the only representation of this algebra is two-dimensional. In fact, this representation shows that the quasi-fermion algebra can be obtained by a (nonlinear) transformation on the usual fermion algebra by

$$a = (1-s\bar{s})^{1/2}c + sc^*c. \tag{23}$$

It can be directly verified that a satisfies (15). The inverse transformation is given by

$$c = (1-s\bar{s})^{-1/2}a - s(1-s\bar{s})^{-1/2}a^*a. \tag{24}$$

As the simplest physical application we can consider a quasi-fermion interacting through a Hamiltonian

$$H = \epsilon a^*a + \lambda(a + a^*). \tag{25}$$

For $s = 0$ this describes an ordinary fermion and the energy levels are perturbed such that to lowest order in λ

$$E_1 = \frac{-\lambda^2}{\epsilon} \qquad E_2 = \epsilon + \frac{\lambda^2}{\epsilon}. \tag{26}$$

For general s, it can be shown that the energy levels are perturbed such that

$$E_{1,2} = \tfrac{1}{2}(1 + 2\lambda s \pm \sqrt{1 + 4\lambda s + 4\lambda^2}) \tag{27}$$

which shows that the perturbation to higher state is of order λ for $s \neq 0$:

$$E_1 = \epsilon + 2\lambda s + \cdots. \tag{28}$$

Hence a quasi-fermion obeying (9) may exhibit measurable differences from an ordinary fermion.

3. The multidimensional and the continuous case

A d-dimensional covariant generalization of this algebra is possible provided that s is generalized to possess d components which transform covariantly under $U(d)$. We propose

$$\begin{aligned} &a_i a_j^* + a_j^* a_i = s_i a_j^* + \bar{s}_j a_i + \delta_{ij} - s_i \bar{s}_j \\ &a_i a_j + a_j a_i = s_i a_j + s_j a_i \qquad s_i \in \mathbb{C} \qquad \sum_i |s_i|^2 < 1 \end{aligned} \tag{29}$$

$i, j = 1, 2, \ldots, d$. This algebra is invariant under the action of the $U(d)$ group which transforms the complex vector s_i together with the operators a_i:

$$a_i \to \sum_j u_{ij} a_j \qquad s_i \to \sum_j u_{ij} s_j. \tag{30}$$

By a unitary transformation we can choose all the s_i to be equal. The subgroup of $U(d)$ which leaves this vector invariant is $U(d-1)$. The permutation group S_d which permutes the a_i is a discrete subgroup of this $U(d-1)$.

To prove that (29) constitutes a deformed fermion algebra we construct a 2^d-dimensional representation of (29). When the parameters $s_i = 0$ the quasi-fermion algebra (29) will reduce to the usual fermion algebra and this representation will reduce to the 2^d-dimensional unique representation of the fermion algebra. First we can perform a $U(d)$ transformation to transform the s_i to a given direction

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_d \end{pmatrix} \to \begin{pmatrix} s \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad s \in \mathbb{R} \tag{31}$$

which at the same time will transform the a_i into b_i.

Since the norm of a vector is invariant under $U(d)$, we can choose

$$s = \sqrt{|s_1|^2 + |s_2|^2 + \cdots + |s_d|^2}. \tag{32}$$

The commutation relations in this basis become

$$\left.\begin{aligned} &b_1 b_1^* + b_1^* b_1 = s(b_1^* + b_1) + 1 - s^2 \\ &b_1^2 = s b_1 \end{aligned}\right. \tag{33a}$$

$$\left.\begin{aligned} &b_1 b_i^* + b_i^* b_1 = s b_i^* \\ &b_1 b_i + b_i b_1 = s b_i \end{aligned}\right. \tag{33b}$$

$$\left.\begin{aligned} &b_i b_j^* + b_j^* b_i = \delta_{ij} \\ &b_i b_j + b_j b_i = 0 \qquad i, j = 2, 3, \ldots, d. \end{aligned}\right. \tag{33c}$$

It follows that the commutation relations satisfied by the b_i, b_i^* among themselves for $i, j = 2, 3, \ldots, d$ (33c) are just standard fermion commutation relations. It can be shown that

$$\begin{aligned} b_1 &= a \otimes I \otimes I \otimes I \otimes \cdots \otimes I \otimes I \\ b_2 &= \rho \otimes c \otimes I \otimes I \otimes \cdots \otimes I \otimes I \\ b_3 &= \rho \otimes \sigma \otimes c \otimes I \otimes \cdots \otimes I \otimes I \\ &\vdots \\ b_d &= \rho \otimes \sigma \otimes \sigma \otimes \sigma \otimes \cdots \otimes \sigma \otimes c \end{aligned} \tag{34}$$

where a, a^* satisfy (15), and c, c^* satisfy (1). ρ satisfies

$$a\rho + \rho a = s\rho \qquad \rho^2 = 1 \qquad \rho = \rho^* \tag{35}$$

and is given by

$$\rho = (1 - s\bar{s})^{-1/2}(aa^* - a^*a). \tag{36}$$

Since (15) requires $|s| < 1$ the s_i have to satisfy that the norm of the vector s_i is less than unity (29). The explicit representations are given by

$$a = \begin{pmatrix} 0 & \sqrt{1-s\bar{s}} \\ 0 & s \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho = \begin{pmatrix} \sqrt{1-s\bar{s}} & s \\ s & -\sqrt{1-s\bar{s}} \end{pmatrix}. \tag{37}$$

The a_i, a_i^* which satisfy (29) are given by a unitary transformation on (34) where

$$s_i = u_{i1}s \qquad a_i = \sum_j u_{ij} b_j. \tag{38}$$

Thus the deformed fermion algebra defined in (29) is consistent. For $d = 2$ this computation gives

$$a_1 = \begin{pmatrix} 0 & -\bar{s}_2 r & s_1 r & -\bar{s}_2 \\ 0 & 0 & 0 & s_1 r \\ 0 & -\bar{s}_2 & s_1 & \bar{s}_2 r \\ 0 & 0 & 0 & s_1 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 & \bar{s}_1 r & s_2 r & \bar{s}_1 \\ 0 & 0 & 0 & s_2 r \\ 0 & \bar{s}_1 & s_2 & -\bar{s}_1 r \\ 0 & 0 & 0 & s_2 \end{pmatrix} \tag{39}$$

where $r = \sqrt{\left(|s_1|^2 + |s_2|^2\right)^{-1} - 1}$.

We now show that (29) has a continuous generalization just as the standard fermion algebra (2) has the continuous generalization (4). We replace the discrete indices i, j by the continuous indices p, q and replace the summation over i by an integral over p. We obtain

$$\begin{aligned} & a(p)a^*(q) + a^*(q)a(p) = s(p)a^*(q) + \overline{s(q)}a(p) + \delta(p-q) - s(p)\overline{s(q)} \\ & a(p)a(q) + a(q)a(p) = s(p)a(q) + s(q)a(p) \\ & \int_{-\infty}^{\infty} |s(p)|^2 \, \mathrm{d}p < 1. \end{aligned} \tag{40}$$

Here $s(p)$ can be called a deformation wavefunction and the case $s(p) \equiv 0$ corresponds to the undeformed fermionic oscillator (4). For the discrete case, the total number operator

$$N = \sum_i a_i^* a_i \tag{41}$$

has integer eigenvalues. This can be shown by using the invariance of N under the unitary group $U(d)$. Performing this transformation gives

$$N = \sum_i b_i^* b_i \tag{42}$$

where b_i^*, b_i satisfy (33). Each $b_i^* b_i$ term has eigenvalue 0 or 1. Moreover these terms commute with each other. Therefore they can be simultaneously diagonalized. This proves that N has integer eigenvalues. It can be shown that just as the $b_i^* b_i$ terms in (42) commute among themselves the $a_i^* a_i$ terms in (41) also do so. Each $a_i^* a_i$ term in (41) has eigenvalues 0, 1. This is because putting $i = j$ in (29) gives the algebra (15) with $a = a_i$ and $s = s_i$.

It is plausible that the limit leading to (40) preserves this property of N having integer eigenvalues just as the limit leading to (4) preserves this property. Thus the total number operator for the continuous case can be defined by

$$N = \int_{-\infty}^{\infty} a^*(p)a(p) \, \mathrm{d}p. \tag{43}$$

Although equations (40) are considerably more complicated compared to (4) it is possible to use this algebra for physical applications. The formalism can easily be generalized to higher dimensions and can be made to include more than one kind of fermion. For d fermions in three dimensions, the algebra is given by

$$a_i(\boldsymbol{p})a_j^*(\boldsymbol{q}) + a_j^*(\boldsymbol{q})a_i(\boldsymbol{p}) = s_i(\boldsymbol{p})a_j^*(\boldsymbol{q}) + \overline{s_j(\boldsymbol{q})}a_i(\boldsymbol{p}) + \delta_{ij}\delta^3(\boldsymbol{p}-\boldsymbol{q}) - s_i(\boldsymbol{p})\overline{s_j(\boldsymbol{q})}$$
$$a_i(\boldsymbol{p})a_j(\boldsymbol{q}) + a_j(\boldsymbol{q})a_i(\boldsymbol{p}) = s_i(\boldsymbol{p})a_j(\boldsymbol{q}) + s_j(\boldsymbol{q})a_i(\boldsymbol{p}),$$
$$\int_{-\infty}^{\infty}\sum_i |s_i(\boldsymbol{p})|^2\,\mathrm{d}^3\boldsymbol{p} < 1 \tag{44}$$
$$N_i = \int_{-\infty}^{\infty} a_i^*(\boldsymbol{p})a_i(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} \qquad N = \sum_i N_i \qquad i = 1, 2, \ldots, d$$

where N_i and N have integer eigenvalues.

4. q-deformation and relation to q-SUSY algebra

In this section, we will present a q-deformation of the one-dimensional generalized fermion algebra (15) and show that this deformation is equivalent to a supercovariant system of q-oscillators [5, 11]. Thus q-deformation, in this example, brings supersymmetrization introducing q-bosons in addition to q-fermions. To achieve this we first define

$$f = s - a \tag{45}$$

where s is now interpreted as a central operator and eliminate s from the algebra (15) including the relations that s and s^* commute with a and a^*. We obtain

$$aa^* + f^*f = a^*a + ff^* = 1 \tag{46}$$
$$af = fa = 0 \tag{47}$$
$$[a, a^*] = [f, f^*] = [a^*, f] = [f^*, a]. \tag{48}$$

A q-deformation of this algebra is given by keeping (46) and (47) but deforming the last two commutators of (48)

$$[a, a^*] = [f, f^*] = qa^*f - q^{-1}fa^* = qf^*a - q^{-1}af^*. \tag{49}$$

To show that this system is equivalent to a SUSY system of oscillators we define

$$A = (a^*a^2 + f^2f^*)/(1-q^2) \tag{50}$$
$$F = ff^*a. \tag{51}$$

The relations (46), (47), (49) show that A, F and their Hermitian conjugates satisfy

$$\begin{aligned} &FF^* + F^*F = 1 - (1-q^2)A^*A \\ &AA^* - q^2A^*A = 1 \\ &AF = qFA \\ &AF^* = qF^*A \\ &F^2 = 0. \end{aligned} \tag{52}$$

This coincides with the one-dimensional supersymmetric algebras in [10, 12]. Its representation [10] is given by the following action on the states $|n_b, n_f\rangle$ where $n_b = 0, 1, 2, \ldots$ is the number of deformed bosons and $n_f = 0, 1$ is the number of deformed fermions

$$
\begin{aligned}
A|n_b, n_f\rangle &= \sqrt{[n_b]}|n_b - 1, n_f\rangle \\
A^*|n_b, n_f\rangle &= \sqrt{[n_b + 1]}|n_b + 1, n_f\rangle \\
F|n_b, 0\rangle &= 0 \\
F|n_b, 1\rangle &= q^{n_b}|n_b, 0\rangle \\
F^*|n_b, 0\rangle &= q^{n_b}|n_b, 1\rangle \\
F^*|n_b, 1\rangle &= 0
\end{aligned}
\tag{53}
$$

where $[n_b] = (1 - q^{2n_b})/(1 - q^2)$. As can be seen from the above relations A^*, A are the deformed boson creation–annihilation operators whereas F^*, F are the deformed fermion creation–annihilation operators.

5. Conclusion

In this paper, we have discussed deformed fermion algebras starting from the relation $a^2 = sa$, our main objective being to construct the discrete and continuous multidimensional extensions similar to that of the standard fermion algebra which corresponds to the case $s = 0$. We have succeeded in doing this, however a question still remains unanswered. Although we were able to q-deform the $s \neq 0$ algebra and show that q-deformation brings supersymmetrization introducing q-bosons, we have not been to able to extend this q-deformation to the multidimensional and continuous cases. We remark that the complex number s, in the process of q-deformation becomes an operator related to the bosonic sector of the q-deformation. Perhaps, a multidimensional q-deformation requires that the parameter q also becomes an operator.

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